Vortex velocity probability distributions in phase-ordering kinetics

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The calculation of the point vortex velocity probability distribution function (VVPDF) is extended to a larger class of systems beyond the nonconserved time-dependent Ginzburg-Landau (TDGL) model treated earlier. The range is extended to include certain anisotropic models and the conserved order parameter case. The VVPDF still satisfies scaling with large velocity tails as for the nonconserved isotropic case. It is shown that the average vortex speed can be self-consistently expressed in terms of correlation functions associated with a Gaussian auxiliary field. In the conserved order parameter case the average vortex speed decays as t^{-1} compared to the $t^{-1/2}$ decay for the nonconserved case.

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I. INTRODUCTION

In recent work [1] it was shown that the theoretically predicted [2] velocity probability distribution for point vortices in the case of a phase-ordering system agrees very well with direct numerical simulations. In particular the predicted high velocity algebraic tail is found to be robust and the predicted exponent confirmed. In the original paper [2] describing the theory there were assumptions concerning the Gaussian nature of an underlying auxiliary field. Here we clarify this result by showing that the assumption of an underlying Gaussian field is consistent and does not imply that the underlying order parameter field is Gaussian. We only require that the order parameter field and the auxiliary field share the same zeros and symmetry. The theory is also extended here to conserved and anisotropic systems of coarsening point defects.

In Ref. [2] it was shown for a nonconserved time-dependent Ginzburg-Landau (TDGL) model that for annihilating point defects $n=d$, where n is the number of components of the order parameter and d the spatial dimensionality of the system, that the vortex velocity probability distribution function (VVPDF), the probability that a given vortex has the velocity $\mathbf{V}$ at time t after a quench, is given by

$$P(\mathbf{V},t)=\frac{\Gamma\left(\frac{n}{2}+1\right)}{[\pi\overline{v}^2(t)]^{n/2}}\frac{1}{(1+\mathbf{V}^2/\overline{v}^2(t))^{(n+2)/2}}, \tag{1}$$

where the parameter $\overline{v}(t)$ is clearly related to the average vortex speed and varies as $t^{-1/2}$ for long times for the nonconserved TDGL model. Both the form of $P(\mathbf{V},t)$ and the time dependence of $\overline{v}(t)$ have been confirmed in Ref. [1] for the case $n=d=2$. It is worth pointing out that the order parameter growth law $L(t)$ for the system studied in Ref. [1] has a log correction [3,4], $L^2\approx t/\ln t$. This log correction for $L(t)$ is seen in the simulations in Ref. [1]. There are no log corrections found for $\overline{v}(t)$. Thus nonlinearities which influence L are not seen in $\overline{v}(t)$. We discuss in more detail here the derivation of the result given by Eq. (1) and its extension to systems with a conserved order parameter and simple spatial anisotropy. In the end we find that the VVPDF still satisfies a form similar to Eq. (1) with the same large velocity tail, but the average vortex speed falls off as t^{-1} in the conserved case compared to the $t^{-1/2}$ behavior found for the nonconserved case.

The set of problems of interest here are driven by Langevin equations of the form

$$\frac{\partial \psi_\alpha}{\partial t}=K_\alpha(\psi), \tag{2}$$

where we assume

$$\lim_{\psi\to 0}K_\alpha(\psi)=-\hat{O}\psi_\alpha, \tag{3}$$

and the right-hand side is linear in ψ. The key idea is that we are interested in that part of the equation of motion which corresponds to the motion of vortex cores which are characterized by zeros of the order parameter. An important example is the TDGL model which is of the form

$$\frac{\partial \psi_\alpha}{\partial t}=-\hat{\Gamma}[V'_\alpha(\psi)-c\nabla^2\psi_\alpha], \tag{4}$$

where $c>0$, $\hat{\Gamma}$ is a constant for a nonconserved order parameter (NCOP) and $\hat{\Gamma}=-D\nabla^2$ for a conserved order parameter (COP). Comparing Eqs. (4) and (3) we have

$$\hat{O}_{NCOP}(1)=-\Gamma c\nabla_1^2 \tag{5}$$

and

$$\hat{O}_{COP}(1)=D\nabla_1^2[-r+c\nabla_1^2] \tag{6}$$

for the COP case. Here $r=V''(\psi)|_{\psi=0}$ and $r<0$ if the system is unstable. Through a proper choice of length and time scales we can choose

$$\hat{O}_{NCOP}(1)=-\nabla_1^2, \tag{7}$$

$$\hat{O}_{COP}(1)=\nabla_1^2[1+\nabla_1^2]. \tag{8}$$

An anisotropic model can be assumed to be of the form

$$\hat{O}_{ANI} = -\sum_{\mu_1} c_{\mu_1}\nabla^2_{\mu_1} + \sum_{\mu_1,\mu_2} b_{\mu_1,\mu_2}\nabla^2_{\mu_1}\nabla^2_{\mu_2}, \tag{9}$$

where c_{μ_1} and b_{μ_1,μ_2} are constants. This reduces to the COP case if $c_{\mu_1}=-1$ and $b_{\mu_1,\mu_2}=1$.

There is the underlying assumption that the nonlinear potential contribution in the equation of motion must be such that system orders via annihilating point defects.

II. DEFECT DENSITIES AND CONTINUITY EQUATION

We assume that the instantaneous positions of these defects are determined by the zeros of the order parameter field. Furthermore, it was pointed out in Ref. [2], that the vortex charge density for this system can be written as $\rho = \delta(\vec{\psi})\mathcal{D}$ where $\mathcal{D}$ is the Jacobian (determinant) for the change of variables from the set of vortex positions $r_i(t)$ (where $\vec{\psi}$ vanishes) to the field $\vec{\psi}$:

$$\mathcal{D} = \frac{1}{n!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\nabla_{\mu_1}\psi_{\nu_1}\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}, \tag{10}$$

where $\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}$ is the n-dimensional fully antisymmetric tensor and summation over repeated indices here and below is implied. Furthermore, since topological charge is conserved, it was shown in Ref. [2] that ρ satisfies a continuity equation:

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot(\rho\mathbf{v}), \tag{11}$$

where the vortex velocity is given by

$$\mathcal{D}v_\beta = -\frac{1}{(n-1)!}\epsilon_{\beta,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}(\hat{O}\psi_{\nu_1})\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}. \tag{12}$$

It is assumed that the velocity field is used inside expressions multiplied by the vortex locating δ function so we can use Eq. (3) in Eq. (12). These results are rather general.

Notice that ρ and $\mathbf{v}$ have certain important invariance properties. If we can write

$$\psi_\nu(1) = [\alpha + \beta m^2(1) + \cdots]m_\nu(1) \tag{13}$$

for small m_ν and where α and β are constants, it is easy to see that $\rho(\vec{\psi})=\rho(\vec{m})$. If we further assume, as $\vec{m}$ and $\vec{\psi}$ go to zero,

$$\hat{O}\psi_\nu(1) = \alpha\hat{O}m_\nu(1), \tag{14}$$

then

$$v_\mu(\vec{\psi}) = v_\mu(\vec{m}). \tag{15}$$

If $\hat{O}$ corresponds to an operator with two gradients, as in the NCOP TDGL model, then, assuming Eq. (13) to be valid, Eq. (14) follows. In the case where $\hat{O}$ has higher-order derivatives and we assume that Eq. (14) holds, then Eq. (13) must be modified.

Thus the correlation function we compute in the next section, $G(12)$, is for that set of fields $\mathbf{m}$, related to ψ by Eq. (13), for small values of m_ν and ψ_ν, which is described by a Gaussian distribution. Thus we assume there is a field m_ν which is Gaussian while the statistics of ψ_ν are largely undetermined.

III. CORRELATIONS FOR THE DEFECT SECTOR

We show in Sec. IV that in determining the average vortex speed we need certain correlation functions for the auxiliary field $\mathbf{m}$, introduced in the last section. We show here that we can use the defect continuity equation (11) to show that there is a self-consistent solution for an $\mathbf{m}$ field that is Gaussian. Furthermore, we determine the correlation functions needed to evaluate the average vortex speed explicitly. The method we develop here is a generalization of the approach due to Mazenko and Wickham [5].

The idea is to look at the equation generated by multiplying the continuity equation by a source function and then averaging over $\mathbf{m}$. We have

$$\langle[\partial\rho(1)\,\partial t_1 + \vec{\nabla}^{(1)}\cdot(\rho(1)\mathbf{v}(1))]S(H)\rangle = 0, \tag{16}$$

where $S(H)=\exp[\int d\bar{1}\mathbf{H}(\bar{1})\cdot\mathbf{m}(\bar{1})]$. The question is whether this equation can be satisfied for an underlying Gaussian probability distribution for arbitrary external field $\mathbf{H}(1)$.

To answer this question we evaluate first the quantity

$$\langle\rho(1)S(H)\rangle = \langle S(H)\delta(1)\mathcal{D}(1)\rangle, \tag{17}$$

where we introduce the simplifying notation $\delta(1)=\delta(\mathbf{m}(1))$ and now $\mathcal{D}$ is a function of the field $\mathbf{m}$. When we talk about correlation in the defect sector we mean averages like in Eq. (17) where there is a vortex locating δ function inside the average.

By taking functional derivatives we are able to generate the correlations between fields $\mathbf{m}$ at arbitrary space-time points with the field at the space-time point 1. If we define

$$Z_H(1) = \langle S(H)\delta(1)\rangle \tag{18}$$

then

$$\langle\delta(1)m_{\nu_2}(2)m_{\nu_3}(3)\cdots\rangle = Z_H^{-1}(1)\frac{\delta}{\delta H_{\nu_2}(2)}\frac{\delta}{\delta H_{\nu_3}(3)}\cdots Z_H(1). \tag{19}$$

In our development a key property of the underlying Gaussian distribution function is

$$\langle m_{\nu_1}(1)A\rangle = \sum_{\nu_{1'}}\int d\bar{1}G_{\nu_1\nu_{1'}}(1\bar{1})\left\langle\frac{\delta}{\delta m_{\nu_{1'}}(\bar{1})}A\right\rangle$$

for arbitrary A. For $A=m_{\nu_2}(2)$ we obtain

$$\langle m_{\nu_1}(1)m_{\nu_2}(2)\rangle = G_{\nu_1\nu_2}(12). \tag{20}$$

If we assume that the system is isotropic in the vector space, then $G_{\nu_1\nu_2}(12)=\delta_{\nu_1\nu_2}G(12)$ and

$$\langle m_{\nu_1}(1)A\rangle = \int d\bar{1}G(1\bar{1}) \left\langle \frac{\delta}{\delta m_{\nu_1}(\bar{1})}A \right\rangle. \tag{21}$$

We then need to work out

$$\langle S(H)\rho(1)\rangle = \left\langle S(H)\delta(1)\frac{1}{n!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \times\nabla_{\mu_1}m_{\nu_1}\nabla_{\mu_2}m_{\nu_2}\cdots\nabla_{\mu_n}m_{\nu_n}\right\rangle. \tag{22}$$

Using Eq. (21) we have

$$\langle S(H)\rho(1)\rangle = \frac{1}{n!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\nabla_{\mu_1}G(1\bar{1}) \times\left\langle \frac{\delta}{\delta m_{\nu_1}(\bar{1})}(S(H)\delta(1)\nabla_{\mu_2}m_{\nu_2}\cdots\nabla_{\mu_n}m_{\nu_n})\right\rangle. \tag{23}$$

The derivative of $S(H)$ leads to the introduction of the quantity

$$A_{\nu_1}(1) = \int d\bar{1}G(1\bar{1})H_{\nu_1}(\bar{1}). \tag{24}$$

We assume, and check self-consistently, that $(\nabla_{\mu_1}G(1\bar{1}))|_{1=\bar{1}}=0$. The derivatives of the product $\nabla_{\mu_2}m_{\nu_2}\cdots\nabla_{\mu_n}m_{\nu_n}$ with respect to $m_{\nu_1}(\bar{1})$ lead to contributions which all vanish because it picks out terms $\delta_{\nu_1\nu_j}$ which multiplies $\epsilon_{\nu_1,\nu_2,.,\nu_j,.,\nu_n}$ and $\epsilon_{\nu_1,\nu_2,.,\nu_1,.,\nu_n}=0$. We have then

$$\langle S(H)\rho(1)\rangle = \frac{1}{n!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\nabla_{\mu_1}A_{\nu_1}(1) \times\langle S(H)\delta(1)\nabla_{\mu_2}m_{\nu_2}\cdots\nabla_{\mu_n}m_{\nu_n}\rangle. \tag{25}$$

Clearly we can go through this process $n-1$ more times to obtain $\langle S(H)\rho(1)\rangle=\mathcal{D}_A(1)Z_H(1)$ where

$$\mathcal{D}_A(1) = \frac{1}{n!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\nabla_{\mu_1}A_{\nu_1}(1) \times\nabla_{\mu_2}A_{\nu_2}(1)\cdots\nabla_{\mu_n}A_{\nu_n}(1). \tag{26}$$

Next we look at the current contributions to Eq. (16) in the form

$$\langle S(H)\vec{\nabla}\cdot(\rho(1)\mathbf{v}(1))\rangle = -\nabla^{(1)}_{\mu_1}\frac{1}{(n-1)!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \times\langle S(H)\delta(1)\dot{m}_{\nu_1}\nabla_{\mu_2}m_{\nu_2}\cdots\nabla_{\mu_n}m_{\nu_n}\rangle. \tag{27}$$

We assume that

$$\delta(1)\dot{m}_{\nu_1}(1) = -\delta(1)\hat{O}(1)m_{\nu_1}(1), \tag{28}$$

where $\hat{O}(1)$ is a derivative operator defined by Eq. (3). In Ref. [2] one has the choice $\hat{O}(1)=-\nabla_1^2$. After inserting Eq. (28) back into Eq. (27) and using Eq. (21) we obtain

$$\langle S(H)\vec{\nabla}\cdot(\rho(1)\mathbf{v}(1))\rangle = -\nabla^{(1)}_{\mu_1}\frac{1}{(n-1)!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}[(-\hat{O}(1)A_{\nu_1}(1)) \times\nabla_{\mu_2}A_{\nu_2}(1)\cdots\nabla_{\mu_n}A_{\nu_n}(1)Z_H(1) + (-\hat{O}(1)G(1\bar{1}))_{1=\bar{1}} \times\langle S(H)\delta_{\nu_1}(1)\nabla_{\mu_2}m_{\nu_2}\cdots\nabla_{\mu_n}m_{\nu_n}\rangle], \tag{29}$$

where $\delta_{\nu_1}(1)=[\partial/\partial m_{\nu_1}(1)]\delta(\mathbf{m}(1))$ and the derivatives of the product of m's vanish as in the previous case. Clearly the reduction of the term containing $\delta_{\nu_1}(1)$ follows just as for $\langle S(H)\rho(1)\rangle$ with the result:

$$\langle S(H)\vec{\nabla}\cdot(\rho(1)\mathbf{v}(1))\rangle = \nabla^{(1)}_{\mu_1}\frac{1}{(n-1)!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \times[(\hat{O}(1)A_{\nu_1}(1))\nabla_{\mu_2}A_{\nu_2}(1)\cdots\nabla_{\mu_n} \times A_{\nu_n}(1)Z_H(1) + (\hat{O}(1)G(1\bar{1}))_{1=\bar{1}}\nabla_{\mu_2} \times A_{\nu_2}(1)\cdots\nabla_{\mu_n}A_{\nu_n}(1)\langle S(H)\delta_{\nu_1}(1)\rangle]. \tag{30}$$

We must next evaluate $Z_H(1)$ and the related quantity $\langle S(H)\delta_{\nu_1}(1)\rangle$. Determination of $Z_H(1)$ involves evaluation of the Gaussian average,

$$Z_H(1) = \langle\delta(\mathbf{m}(1))S(\mathbf{H})\rangle = \int\frac{d^dk}{(2\pi)^d}\langle e^{i\mathbf{k}\cdot\mathbf{m}(1)}e^{\mathbf{H}(\bar{1})\cdot\mathbf{m}(\bar{1})}\rangle, \tag{31}$$

with the result

$$Z_H(1) = \frac{e^{-(1/2)[A^2(1)/S_0(1)]}}{(2\pi S_0(1))^{n/2}}\exp\left[\frac{1}{2}H_{\nu_1}(\bar{1})H_{\nu_1}(\bar{2})G(\overline{12})\right], \tag{32}$$

where $S_0(1)=G(11)$. Next we need

$$\begin{aligned}\langle S(H)\delta_{\nu_1}(1)\rangle &= \int\frac{d^dk}{(2\pi)^d}ik_{\nu_1}\langle e^{i\mathbf{k}\cdot\mathbf{m}(1)}e^{\mathbf{H}(\bar{1})\cdot\mathbf{m}(\bar{1})}\rangle\\ &= \int\frac{d^dk}{(2\pi)^d}\, ik_{\nu_1}\exp\left[-\frac{1}{2}k^2G(11)+i\mathbf{k}\cdot\mathbf{A}(1) + \frac{1}{2}H_{\nu_2}(\bar{1})H_{\nu_2}(\bar{2})G(\overline{12})\right]\\ &= \exp\left[\frac{1}{2}H_{\nu_1}(\bar{1})H_{\nu_1}(\bar{2})G(\overline{12})\right]\frac{\partial}{\partial A_{\nu_1}(1)} \times\exp\left[-\frac{1}{2}H_{\nu_1}(\bar{1})H_{\nu_1}(\bar{2})G(\overline{12})\right]Z_H(1)\\ &= -\frac{A_{\nu_1}(1)}{S_0(1)}Z_H(1).\end{aligned} \tag{33}$$

Putting Eqs. (32) and (33) back into Eq. (30) and allowing the gradient $\nabla^{(1)}_{\mu_1}$ to act gives

$$\langle S(H)\vec{\nabla}\cdot(\rho(1)\mathbf{v}(1))\rangle = -\mathcal{D}_B(1)Z_H(1) - \mathcal{D}^B_{\mu_1}(1)\nabla^{(1)}_{\mu_1}Z_H(1), \tag{34}$$

where

$$\mathcal{D}_B(1) = \frac{1}{(n-1)!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \times\nabla_{\mu_1}B_{\nu_1}(1)\nabla_{\mu_2}A_{\nu_2}(1)\cdots\nabla_{\mu_n}A_{\nu_n}(1), \tag{35}$$

and

$$\mathcal{D}^B_{\mu_1}(1) = \frac{1}{(n-1)!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}B_{\nu_1}(1) \times\nabla_{\mu_2}A_{\nu_2}(1)\cdots\nabla_{\mu_n}A_{\nu_n}(1). \tag{36}$$

$$B_{\nu_1}(1) = (-\hat{O}(1) + \Omega(1))A_{\nu_1}(1) \tag{37}$$

and

$$\Omega(1) = \frac{1}{S_0(1)}(\hat{O}(1)G(12))_{1=2}. \tag{38}$$

Putting the results together in Eq. (16) we obtain

$$\frac{\partial}{\partial t_1}(\mathcal{D}_A(1)Z_H(1)) = \mathcal{D}_B(1)Z_H(1) + \mathcal{D}^B_{\mu_1}(1)\nabla^{(1)}_{\mu_1}Z_H(1). \tag{39}$$

We can write this in the form

$$\frac{\partial\mathcal{D}_A(1)}{\partial t_1} + \mathcal{D}_A(1)\frac{\partial}{\partial t_1}\ln Z_H(1) = \mathcal{D}_B(1) + \mathcal{D}^B_{\mu_1}(1)\nabla_{\mu_1}\ln Z_H(1). \tag{40}$$

After taking the derivatives $Z_H(1)$, we can then write Eq. (40) in the form

$$W_2(1) = W_4(1), \tag{41}$$

where

$$W_2(1) = \frac{\partial\mathcal{D}_A(1)}{\partial t_1} - \mathcal{D}_A(1)\frac{n}{2}\frac{\dot{S}_0(1)}{S_0(1)} - \mathcal{D}_B(1) \tag{42}$$

and

$$W_4(1) = \mathcal{D}_A(1)\left(\frac{\mathbf{A}(1)}{S_0(1)}\cdot\dot{\mathbf{A}}(1) - \frac{1}{2}A^2(1)\frac{\dot{S}_0(1)}{S_0^2(1)}\right) - \mathcal{D}^B_{\mu_1}(1)\frac{\mathbf{A}(1)}{S_0(1)}\cdot\nabla_{\mu_1}\mathbf{A}(1). \tag{43}$$

Look first at $W_2(1)$ which can be written in the form

$$\begin{aligned} W_2(1) &= \frac{1}{(n-1)!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \times\nabla_{\mu_1}\left(\dot{A}_{\nu_1}(1) - \frac{1}{2}\frac{\dot{S}_0(1)}{S_0(1)}A_{\nu_1}(1) - B_{\nu_1}(1)\right) \times\nabla_{\mu_2}A_{\nu_2}(1)\cdots\nabla_{\mu_n}A_{\nu_n}(1) \\ &= \frac{1}{(n-1)!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \times\nabla_{\mu_1}g_{\nu_1}(1)\nabla_{\mu_2}A_{\nu_2}(1)\cdots\nabla_{\mu_n}A_{\nu_n}(1), \end{aligned} \tag{44}$$

where

$$g_{\nu_1}(1) = \dot{A}_{\nu_1}(1) - \frac{1}{2}\frac{\dot{S}_0(1)}{S_0(1)}A_{\nu_1}(1) - B_{\nu_1}(1). \tag{45}$$

In looking at W_4 we need to focus on the quantity

$$\mathcal{D}^B_{\mu_1}(1)A_\nu(1)\nabla_{\mu_1}A_\nu(1) = \frac{1}{(n-1)!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}B_{\nu_1}(1) \times\nabla_{\mu_2}A_{\nu_2}(1)\cdots\nabla_{\mu_n}A_{\nu_n}(1)A_\nu(1)\nabla_{\mu_1}A_\nu(1). \tag{46}$$

Note that

$$\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\nabla_{\mu_1}A_\nu(1)\nabla_{\mu_2}A_{\nu_2}(1)\cdots\nabla_{\mu_n}A_{\nu_n}(1) = \epsilon_{\nu,\nu_2,\ldots,\nu_n}\mathcal{D}_A(1). \tag{47}$$

Putting this back into Eq. (46), we find

$$\begin{aligned} \mathcal{D}^B_{\mu_1}(1)A_\nu(1)\nabla_{\mu_1}A_\nu(1) &= \frac{1}{(n-1)!}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\epsilon_{\nu,\nu_2,\ldots,\nu_n}B_{\nu_1}(1)A_\nu(1)\mathcal{D}_A(1) \\ &= B_{\nu_1}(1)A_{\nu_1}(1)\mathcal{D}_A(1) \end{aligned} \tag{48}$$

and

$$\begin{aligned} W_4(1) &= \mathcal{D}_A(1)\frac{A_\nu(1)}{S_0(1)}\left(\dot{A}_{\nu_1}(1) - \frac{1}{2}\frac{\dot{S}_0(1)}{S_0(1)}A_{\nu_1}(1) - B_{\nu_1}(1)\right) \\ &= \mathcal{D}_A(1)\frac{A_\nu(1)}{S_0(1)}g_\nu(1). \end{aligned} \tag{49}$$

Using Eqs. (44) and (49) in Eq. (41) we find a solution for general **H** if

$$g_\nu(1) = \dot{A}_\nu(1) - \frac{1}{2}\frac{\dot{S}_0(1)}{S_0(1)}A_\nu(1) - B_\nu(1) = 0. \tag{50}$$

This will hold for all source fields **H** if

$$\frac{\partial}{\partial t_1}G(12) - \frac{1}{2}\frac{\dot{S}_0(1)}{S_0(1)}G(12) = [-\hat{O}(1) + \Omega(1)]G(12). \tag{51}$$

Thus the average of the continuity equation (16) is satisfied by a Gaussian probability distribution if the associated variance $G(12)$ satisfies Eq. (51).

IV. SOLUTION FOR $G(12)$

We can solve Eq. (51) for $G(12)$ in some generality. The first step is to write

$$G(12) = \sqrt{S_0(1)S_0(2)}f(12), \tag{52}$$

where $S_0(1)=G(11)$. Inserting this form into Eq. (51) gives

$$\frac{\partial}{\partial t_1}f(12) = [-\hat{O}(1) + \Omega(1)]f(12). \tag{53}$$

We assume that the system is translationally invariant and on Fourier transformation the operator $\hat{O}(1)$ is *diagonalized* and time independent. We have then

$$\frac{\partial}{\partial t_1}f(q,t_1,t_2) = [-O(q) + \Omega(1)]f(q,t_1,t_2). \tag{54}$$

We see that $\Omega(1)$, defined by Eq. (38), is determined by the constraint

$$\Omega(1) = \int \frac{d^dq}{(2\pi)^d}\, O(q)f(q,t_1,t_1). \tag{55}$$

We also have the equation

$$\frac{\partial}{\partial t_2}f(q,t_1,t_2) = [-O(q) + \Omega(2)]f(q,t_1,t_2). \tag{56}$$

Adding Eqs. (54) and (56) and setting $t_1=t_2=t$ we obtain for the equal-time correlation function $f(q,t)\equiv f(q,t,t)$:

$$\frac{\partial}{\partial t}f(q,t) = 2[-O(q) + \Omega(2)]f(q,t). \tag{57}$$

For equal times we have from Eq. (52) that $f(11)=1$ or

$$1 = \int \frac{d^dq}{(2\pi)^d}f(q,t). \tag{58}$$

The partial solution for Eq. (57) is given by

$$\begin{aligned} f(q,t) &= \exp\left(2\int_{t_0}^{t} d\tau[\Omega(\tau) - O(q)]\right)f(q,t_0) \\ &= R^2(t,t_0)e^{-2O(q)(t-t_0)}f(q,t_0), \end{aligned} \tag{59}$$

where

$$R(t_1,t_2) = \exp\left(\int_{t_2}^{t_1} d\tau\, \Omega(\tau)\right). \tag{60}$$

We then need to solve for $\Omega(t)$. Inserting Eq. (59) into Eq. (58) gives

$$1 = R^2(t,t_0)I(t,t_0), \tag{61}$$

where

$$I(t,t_0) = \int \frac{d^dq}{(2\pi)^d}\, e^{-2O(q)(t-t_0)}f(q,t_0). \tag{62}$$

We then have from Eq. (61)

$$R^2(t,t_0) = I^{-1}(t,t_0). \tag{63}$$

The constraint condition, Eq. (55), is given by

$$\Omega(t) = R^2(t,t_0)\int \frac{d^dq}{(2\pi)^d}O(q)e^{-2O(q)(t-t_0)}f(q,t_0) = -\frac{1}{2}\frac{\dot{I}(t,t_0)}{I(t,t_0)}. \tag{64}$$

Thus the determination of $\Omega(1)$ is reduced to evaluation of the integral $I(t,t_0)$. The equal time correlation function is given then by

$$f(q,t) = I^{-1}(t,t_0)e^{-2O(q)(t-t_0)}f(q,t_0). \tag{65}$$

Going back to the unequal time correlation function we can integrate Eq. (54),

$$\begin{aligned} f(q,t_1,t_2) &= \exp\left(\int_{t_2}^{t_1} d\tau[\Omega(\tau) - O(q)]\right)f(q,t_2) \\ &= R(t_1,t_2)e^{-O(q)(t_1-t_2)}f(q,t_2) \\ &= R(t_1,t_0)R(t_2,t_0)e^{-O(q)(t_1+t_2-2t_0)}f(q,t_0), \end{aligned} \tag{66}$$

where we have used $R(t_1,t_2)=R(t_1,t_0)/R(t_2,t_0)$. We obtain a complete solution once one does the integral for $I(t,t_0)$. Notice that these results are independent of the specific form for $S_0(t)$.

V. EVALUATION OF VORTEX VELOCITY PROBABILITY DISTRIBUTION

The vortex velocity probability distribution function defined by

$$n_0P(\mathbf{V},t) \equiv \langle n(1)\,\delta(\mathbf{V} - \mathbf{v}(1))\rangle, \tag{67}$$

where $\mathbf{v}$ as a function of the order parameter is given by Eq. (12) with $\vec{\psi}$ replaced by $\vec{m}$, $n(1)=\delta(\mathbf{m}(1))|\mathcal{D}(\mathbf{m}(1))|$ is the unsigned defect density, and $n_0(1)=\langle n(1)\rangle$. We notice in evaluating $P(\mathbf{V},t)$ that it is of the *defect sector form*, thus there is a defect locating δ function in the average via the factor of $n(1)$. Our results from Sec. II suggest that in this sector we can treat the field $\mathbf{m}$ as Gaussian with variance given by $G(12)$ calculated in Sec. III.

In carrying out the average we need the auxiliary quantity

$$W(\xi,\mathbf{b}) = \langle\,\delta(\mathbf{m})\prod_{\mu,\nu}\delta(\xi_\mu^\nu - \nabla_\mu m_\nu)\,\delta(\mathbf{b} - \mathbf{K})\rangle, \tag{68}$$

where $\mathbf{K}(1)=\hat{O}(1)\mathbf{m}(1)$. Then

$$n_0 P(\mathbf{V},t) = \int d^n b \prod_{\mu,\nu} d\xi_\mu^\nu |\mathcal{D}(\xi)| \delta(\mathbf{V} - \mathbf{v}(\mathbf{b},\xi)) W(\xi,\mathbf{b}), \tag{69}$$

where $\mathbf{v}(\mathbf{b},\xi) = \mathbf{J}(\mathbf{b},\xi)/\mathcal{D}(\xi)$ with

$$\mathcal{D}(\xi) = \frac{1}{n!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \xi_{\mu_1}^{\nu_1} \xi_{\mu_2}^{\nu_2} \cdots \xi_{\mu_n}^{\nu_n} \tag{70}$$

and

$$J_\alpha(\mathbf{b},\xi) = \frac{1}{(n-1)!} \epsilon_{\alpha,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} b_{\nu_1} \xi_{\mu_2}^{\nu_2} \cdots \xi_{\mu_n}^{\nu_n}. \tag{71}$$

Let us turn to the Gaussian average determining $W(\xi,\mathbf{b})$. Following the analysis used to evaluate Z_H in Sec. IIII we introduce the integral representation for the δ function to obtain

$$W(\xi,\mathbf{b}) = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \left(\prod_{\mu,\nu} \int \frac{ds_\mu^\nu}{2\pi} \right) e^{-i\mathbf{b}\cdot\mathbf{q}} e^{-i\xi_\mu^\nu s_\mu^\nu} \Gamma(\mathbf{k},\mathbf{q},s), \tag{72}$$

where

$$\Gamma(\mathbf{k},\mathbf{q},s) = \langle e^{i\mathbf{k}\cdot\mathbf{m}(1)} e^{i\mathbf{q}\cdot\mathbf{K}(1)} e^{is_\mu^\nu \nabla_\mu m_\nu(1)} \rangle. \tag{73}$$

If we introduce

$$H_\alpha(\bar{1}) = i[k_\alpha + q_\alpha \hat{O}(1) + s_\mu^\alpha \nabla_\mu^{(1)}] \delta(1\bar{1}) \tag{74}$$

then we can write

$$\Gamma(\mathbf{k},\mathbf{q},s) = \langle e^{H_\alpha(\bar{1}) m_\alpha(\bar{1})} \rangle = \exp\left[\frac{1}{2} H_\alpha(\bar{1}) H_\alpha(\bar{2}) G(\bar{1}\bar{2}) \right], \tag{75}$$

where $G(12)$ was determined in Sec. III. We assume that the cross terms involving an odd number of gradients vanishes in the argument of the exponential. We have then

$$\Gamma(\mathbf{k},\mathbf{q},s) = \exp\Big[-\frac{1}{2} [k^2 S_0(1) + 2\mathbf{k}\cdot\mathbf{q} S_c(1) + q^2 S_{O2}(1) + s_\mu^\alpha s_\mu^\alpha S_\mu^{(2)}(1)] \Big], \tag{76}$$

where

$$\frac{S_c(1)}{S_0(1)} = \int \frac{d^n q}{(2\pi)^n} O(q) f(q,t) = \Omega(1) = -\frac{1}{2} \frac{\dot{I}(1)}{I(1)}, \tag{77}$$

where we have used Eq. (54),

$$\frac{S_{O2}(1)}{S_0(1)} = \int \frac{d^n q}{(2\pi)^n} O^2(q) f(q,t) = \frac{1}{4} \frac{\ddot{I}(1)}{I(1)} \tag{78}$$

having used Eq. (65), and

$$\frac{S_{\mu\mu'}^{(2)}(1)}{S_0(1)} = \int \frac{d^n q}{(2\pi)^n} q_\mu q_{\mu'} f(q,t_1) = \delta_{\mu\mu'} \frac{S_\mu^{(2)}(1)}{S_0(1)}. \tag{79}$$

The next step in extracting W is to integrate over $\mathbf{k}$:

$$\Gamma(\mathbf{q},s) = \int \frac{d^n k}{(2\pi)^n} \Gamma(\mathbf{k},\mathbf{q},s) = \frac{1}{(2\pi S_0)^{n/2}} e^{-(1/2)q^2 \bar{S}} e^{-(1/2)s_\mu^\alpha s_\mu^\alpha S_\mu^{(2)}}, \tag{80}$$

where

$$\bar{S} = S_{O2} - \frac{S_c^2}{S_0}. \tag{81}$$

Using Eq. (80) back in Eq. (72) we obtain

$$\begin{aligned} W(\xi,\mathbf{b}) &= \int \frac{d^n q}{(2\pi)^n} \left(\prod_{\mu,\nu} \int \frac{ds_\mu^\nu}{2\pi} \right) e^{-i\mathbf{b}\cdot\mathbf{q}} e^{-i\xi_\mu^\nu s_\mu^\nu} \Gamma(\mathbf{q},s) \\ &= \int \frac{d^n q}{(2\pi)^n} \left(\prod_{\mu,\nu} \int \frac{ds_\mu^\nu}{2\pi} \right) e^{-i\mathbf{b}\cdot\mathbf{q}} e^{-i\xi_\mu^\nu s_\mu^\nu} \frac{1}{(2\pi S_0)^{n/2}} \\ &\quad \times e^{-(1/2)q^2 \bar{S}} e^{-(1/2)s_\mu^\alpha s_\mu^\alpha S_\mu^{(2)}}. \end{aligned} \tag{82}$$

This factorizes into a product of three natural parts,

$$W(\xi,\mathbf{b}) = \frac{1}{(2\pi S_0)^{n/2}} W(\xi) W(\mathbf{b}), \tag{83}$$

where

$$W(\xi) = \left(\prod_{\mu,\nu} \int \frac{ds_\mu^\nu}{2\pi} \right) e^{-i\xi_\mu^\nu s_\mu^\nu} e^{-(1/2)s_\mu^\alpha s_\mu^\alpha S_\mu^{(2)}} \tag{84}$$

and

$$W(\mathbf{b}) = \int \frac{d^n q}{(2\pi)^n} e^{-i\mathbf{b}\cdot\mathbf{q}} e^{-(1/2)q^2 \bar{S}}. \tag{85}$$

Using the basic integral

$$\int \frac{dx}{2\pi} e^{-iyx} e^{-(a/2)x^2} = \frac{1}{\sqrt{2\pi a}} e^{-y^2/2a} \tag{86}$$

we can evaluate both factors:

$$W(\xi) = \left(\prod_\mu \frac{1}{2\pi S_\mu^{(2)}} \right)^{n/2} e^{-(\xi_{\mu'}^\nu)^2/2S_{\mu'}^{(2)}}, \tag{87}$$

$$W(\mathbf{b}) = \frac{1}{(2\pi\bar{S})^{n/2}} e^{-b^2/2\bar{S}}. \tag{88}$$

Turning to $n_0 P(\mathbf{V},t)$ given by Eq. (69), we see that we have the integral over $\mathbf{b}$ of the form

$$\begin{aligned} J_b &= \int d^n b\, \delta(\mathbf{V} - \mathbf{v}(\mathbf{b},\xi)) W(\mathbf{b}) \\ &= \int d^n b \int \frac{d^n z}{(2\pi)^n} e^{-i\mathbf{V}\cdot\mathbf{z}} e^{i\mathbf{v}(\mathbf{b},\xi)\cdot\mathbf{z}} W(\mathbf{b}). \end{aligned} \tag{89}$$

We can then write $\mathbf{z}\cdot\mathbf{v}(\mathbf{b},\xi) = \mathbf{a}\cdot\mathbf{b}$ where

$$a_{\nu_1}=\frac{1}{\mathcal{D}(n-1)!}z_\alpha\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\xi^{\nu_2}_{\mu_2}\cdots\xi^{\nu_n}_{\mu_n} \tag{90}$$

then

$$J_b=\int d^nb\int\frac{d^nz}{(2\pi)^n}\,e^{-i\mathbf{V}\cdot\mathbf{z}}e^{i\mathbf{a}\cdot\mathbf{b}}\frac{1}{(2\pi\bar{S})^{n/2}}e^{-b^2/2\bar{S}}$$
$$=\int\frac{d^nz}{(2\pi)^n}\,e^{-i\mathbf{V}\cdot\mathbf{z}}e^{-(\bar{S}/2)\mathbf{a}^2}, \tag{91}$$

where $\mathbf{a}^2=z_\alpha M_{\alpha\beta}z_\beta$ and the matrix M is given by

$$M_{\alpha,\beta}=\frac{1}{\mathcal{D}^2[(n-1)!]^2}\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\epsilon_{\nu,\nu_2,\ldots,\nu_n}\xi^{\nu_2}_{\mu_2}\cdots\xi^{\nu_n}_{\mu_n}$$
$$\times\epsilon_{\beta,\mu'_2,\ldots,\mu'_n}\epsilon_{\nu,\nu'_2,\ldots,\nu'_n}\xi^{\nu'_2}_{\mu'_2}\cdots\xi^{\nu'_n}_{\mu'_n}. \tag{92}$$

Doing the remaining Gaussian z integration in Eq. (91) we obtain

$$J_b=\frac{1}{(2\pi\bar{S})^{n/2}}\frac{1}{\sqrt{\det M}}\exp\left[-\frac{1}{2\bar{S}}\sum_{\mu,\nu}V^\mu[M^{-1}]_{\mu,\nu}V^\nu\right] \tag{93}$$

and

$$n_0P(\mathbf{V},t)=\int\prod_{\mu,\nu}d\xi^\nu_\mu|\mathcal{D}(\xi)|W(\xi)\frac{1}{(4\pi^2S_0\bar{S})^{n/2}}\frac{1}{\sqrt{\det M}}$$
$$\times\exp\left[-\frac{1}{2\bar{S}}\sum_{\mu,\nu}V^\mu[M^{-1}]_{\mu,\nu}V^\nu\right]. \tag{94}$$

We must look at the matrix M and its inverse. First multiply $M_{\alpha\beta}$ by ξ^ν_α to obtain

$$\xi^\nu_\alpha M_{\alpha\beta}=\xi^\nu_\alpha\frac{1}{\mathcal{D}^2[(n-1)!]^2}\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\xi^{\nu_2}_{\mu_2}\cdots\xi^{\nu_n}_{\mu_n}$$
$$\times\epsilon_{\beta,\mu'_2,\ldots,\mu'_n}\epsilon_{\nu_1,\nu'_2,\ldots,\nu'_n}\xi^{\nu'_2}_{\mu'_2}\cdots\xi^{\nu'_n}_{\mu'_n}. \tag{95}$$

However,

$$\xi^\nu_{\mu_1}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\xi^{\nu_2}_{\mu_2}\cdots\xi^{\nu_n}_{\mu_n}=\mathcal{D}(\xi)\epsilon_{\nu,\nu_2,\ldots,\nu_n} \tag{96}$$

then

$$\xi^\nu_\alpha M_{\alpha\beta}=\frac{1}{\mathcal{D}^2[(n-1)!]^2}\mathcal{D}(\xi)\epsilon_{\nu,\nu_2,\ldots,\nu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\epsilon_{\beta,\mu'_2,\ldots,\mu'_n}$$
$$\times\epsilon_{\nu_1,\nu'_2,\ldots,\nu'_n}\xi^{\nu'_2}_{\mu'_2}\cdots\xi^{\nu'_n}_{\mu'_n} \tag{97}$$

$$=\frac{1}{\mathcal{D}(n-1)!}\epsilon_{\beta,\mu'_2,\ldots,\mu'_n}\epsilon_{\nu,\nu'_2,\ldots,\nu'_n}\xi^{\nu'_2}_{\mu'_2}\cdots\xi^{\nu'_n}_{\mu'_n}, \tag{98}$$

where we have used

$$\epsilon_{\nu,\nu_2,\ldots,\nu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}=\delta_{\nu,\nu_1}(n-1)\,!\,. \tag{99}$$

Multiply Eq. (98) by ξ^ν_γ to obtain

$$\xi^\nu_\gamma\xi^\nu_\alpha M_{\alpha\beta}=\frac{1}{\mathcal{D}(n-1)!}\xi^\nu_\gamma\epsilon_{\beta,\mu'_2,\ldots,\mu'_n}\epsilon_{\nu,\nu'_2,\ldots,\nu'_n}\xi^{\nu'_2}_{\mu'_2}\cdots\xi^{\nu'_n}_{\mu'_n}$$
$$=\frac{1}{\mathcal{D}(n-1)!}\epsilon_{\beta,\mu'_2,\ldots,\mu'_n}\mathcal{D}(\xi)\epsilon_{\gamma,\mu'_2,\ldots,\mu'_n}=\delta_{\gamma\beta}. \tag{100}$$

Thus we have the beautiful result:

$$(M^{-1})_{\alpha\beta}=\sum_\nu\xi^\nu_\alpha\xi^\nu_\beta. \tag{101}$$

We need $\det M=1/\det M^{-1}$. We have

$$\det M^{-1}=\frac{1}{n!}\epsilon_{\alpha_1,\alpha_2,\ldots,\alpha_n}\epsilon_{\beta_1,\beta_2,\ldots,\beta_n}\xi^{\nu_1}_{\alpha_1}\xi^{\nu_1}_{\beta_1}\xi^{\nu_2}_{\alpha_2}\xi^{\nu_2}_{\beta_2}\cdots\xi^{\nu_n}_{\alpha_n}\xi^{\nu_n}_{\beta_n}$$
$$=\frac{1}{n!}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\mathcal{D}(\xi)\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\mathcal{D}(\xi)$$
$$=\frac{1}{n!}\mathcal{D}(\xi)^2n\,!\;=\mathcal{D}(\xi)^2. \tag{102}$$

Using the clean result $\det(M)=1/(\mathcal{D})^2$, and Eq. (101), we have

$$n_0P(\mathbf{V},t)=\int\left(\prod_{\mu,\nu}\frac{d\xi^\nu_\mu}{\sqrt{(2\pi S^{(2)}_\mu)}}\right)\frac{\mathcal{D}^2(\xi)}{(4\pi^2S_0\bar{S})^{n/2}}e^{-(1/2)A(\xi)}, \tag{103}$$

where

$$A(\xi)=\sum_{\mu,\nu}\frac{1}{S^{(2)}_\mu}(\xi^\nu_\mu)^2+\frac{1}{\bar{S}}\sum_{\alpha,\beta,\nu}V^\alpha\xi^\nu_\alpha\xi^\nu_\beta V^\beta. \tag{104}$$

Next make the change of variables $\xi^\nu_\mu=\sqrt{S^{(2)}_\mu}\tilde{\xi}^\mu_\nu$ to obtain, $\mathcal{D}(\xi)=(\prod_\mu\sqrt{S^{(2)}_\mu})\mathcal{D}(\tilde{\xi})$,

$$n_0P(\mathbf{V},t)=\prod_\mu\left(\frac{S^{(2)}_\mu}{2\pi\sqrt{S_0\bar{S}}}\right)\int\left(\prod_{\mu,\nu}\frac{d\tilde{\xi}^\mu_\nu}{\sqrt{(2\pi)}}\right)\mathcal{D}^2(\tilde{\xi})e^{-(1/2)A(\tilde{\xi})}, \tag{105}$$

where

$$A(\tilde{\xi})=\sum_{\mu,\nu}(\tilde{\xi}^\mu_\nu)^2+\sum_{\alpha,\beta,\nu}\tilde{V}^\alpha\tilde{\xi}^\nu_\alpha\tilde{\xi}^\nu_\beta\tilde{V}^\beta \tag{106}$$

and

$$\tilde{V}^\alpha=\sqrt{\frac{S^{(2)}_\alpha}{\bar{S}}}V^\alpha. \tag{107}$$

Next we make the transformation from $\tilde{\xi}^\alpha_\nu$ to χ^ν_α via

$$\tilde{\xi}^\nu_\alpha=N_{\alpha,\beta}\chi^\nu_\beta \tag{108}$$

such that

$$A(\tilde{\xi}) = \sum_{\alpha,\nu} (\chi_\alpha^\nu)^2 = \sum_{\mu,\nu} N_{\mu,\bar{\mu}_1} N_{\mu,\bar{\mu}_2} \chi_{\bar{\mu}_1}^\nu \chi_{\bar{\mu}_2}^\nu + \sum_{\alpha,\beta,\nu} \tilde{V}^\alpha N_{\alpha,\bar{\mu}_1} \chi_{\bar{\mu}_1}^\nu N_{\beta,\bar{\mu}_2} \chi_{\bar{\mu}_2}^\nu \tilde{V}^\beta. \quad (109)$$

This requires that N satisfy

$$N_{\mu,\mu_1} N_{\mu,\mu_2} + \tilde{V}^\alpha N_{\alpha,\mu_1} \tilde{V}^\beta N_{\beta,\mu_2} = \delta_{\mu_1,\mu_2}. \quad (110)$$

This has a solution given by

$$N_{\alpha\beta} = \delta_{\alpha\beta} + \left(\frac{1}{\sqrt{1+\tilde{V}^2}} - 1\right)\hat{V}^\alpha \hat{V}^\beta. \quad (111)$$

We then need the Jacobian of the transformation $d\xi_\mu^\nu \to d\chi_\mu^\nu$ and $\mathcal{D}(\tilde{\xi})$ evaluated in terms of χ. Look first at $\mathcal{D}(\tilde{\xi})$,

$$\mathcal{D}(\tilde{\xi}) = \frac{1}{n!}\epsilon_{\alpha_1\alpha_2\cdots\alpha_n}\epsilon_{\nu_1\nu_2\cdots\nu_n} \times [\chi_{\alpha_1}^{\nu_1} + N_1 \tilde{V}^{\alpha_1}\chi_{\mu_1}^{\nu_1}\tilde{V}^{\bar{\mu}_1}] \times [\chi_{\alpha_2}^{\nu_2} + N_1 \tilde{V}^{\alpha_2}\chi_{\mu_2}^{\nu_2}\tilde{V}^{\bar{\mu}_2}] \cdots [\chi_{\alpha_n}^{\nu_n} + N_1 \tilde{V}^{\alpha_n}\chi_{\mu_n}^{\nu_n}\tilde{V}^{\bar{\mu}_n}]. \quad (112)$$

If we multiply this out in powers of $\tilde{\mathbf{V}}$ we see that if we have more than one factor of $\tilde{V}^{\alpha_i}$ then the contribution vanishes due to antisymmetry, therefore

$$\mathcal{D}(\tilde{\xi}) = \mathcal{D}(\chi) + \frac{1}{n!} n \epsilon_{\alpha_1\alpha_2\cdots\alpha_n}\epsilon_{\nu_1\nu_2\cdots\nu_n} N_1 \tilde{V}^{\alpha_1}\chi_{\mu_1}^{\nu_1}\tilde{V}^{\bar{\mu}_1}\chi_{\alpha_2}^{\nu_2}\cdots\chi_{\alpha_n}^{\nu_n} \quad (113)$$

which can be put in the form

$$\mathcal{D}(\tilde{\xi}) = \frac{\mathcal{D}(\chi)}{\sqrt{1+\tilde{V}^2}}. \quad (114)$$

Next we need the Jacobian

$$J = \prod_\nu \det\left(\frac{\partial \tilde{\xi}_\mu^\nu}{\partial \chi_{\mu'}^\nu}\right) \quad (115)$$

$$= [\det(\delta_{\mu,\mu'} + N_1 \tilde{V}^\mu \tilde{V}^{\mu'})]^n = (J_0)^n, \quad (116)$$

where

$$J_0 = \det(\delta_{\mu,\mu'} + N_1 \tilde{V}^\mu \tilde{V}^{\mu'}) \quad (117)$$

$$= \frac{1}{n!}\epsilon_{\alpha_1\alpha_2\cdots\alpha_n}\epsilon_{\beta_1\beta_2\cdots\beta_n}[\delta_{\alpha_1,\beta_1} + N_1\tilde{V}^{\alpha_1}\tilde{V}^{\beta_1}] \times [\delta_{\alpha_2,\beta_2} + N_1\tilde{V}^{\alpha_2}\tilde{V}^{\beta_2}]\cdots[\delta_{\alpha_3,\beta_3} + N_1\tilde{V}^{\alpha_3}\tilde{V}^{\beta_3}]. \quad (118)$$

Again, expanding this out in powers of $\tilde{V}$, only the first two terms contribute due to symmetry and we have

$$J_0 = \frac{1}{n!}\epsilon^2_{\alpha_1\alpha_2\cdots\alpha_n} + nN_1\epsilon_{\alpha_1\alpha_2\cdots\alpha_n}\epsilon_{\beta_1\alpha_2\cdots\alpha_n}\tilde{V}^{\alpha_1}\tilde{V}^{\beta_1} = 1 + N_1\tilde{V}^2 = \frac{1}{\sqrt{1+\tilde{V}^2}}. \quad (119)$$

Going back to Eq. (105) we have

$$n_0 P[\mathbf{V},t] = \prod_\mu \left(\frac{S_\mu^{(2)}}{2\pi\sqrt{S_0\bar{S}}}\right)\frac{1}{(1+\tilde{V}^2)^{(n+2)/2}} J_F, \quad (120)$$

where we have the final integral

$$J_F = \int \prod_{\mu,\nu} \frac{d\chi_\mu^\nu}{\sqrt{2\pi}}\mathcal{D}^2(\chi)e^{-(1/2)A(\chi)}. \quad (121)$$

We can evaluate J_F directly. The first step is to write

$$J_F = \int \prod_{\mu,\nu}\frac{d\chi_\mu^\nu}{\sqrt{2\pi}}\epsilon_{\mu_1\mu_2\ldots\mu_n}\epsilon_{\mu_1'\mu_2'\ldots\mu_n'} \times \chi_{\mu_1}^{(1)}\chi_{\mu_2}^{(2)}\cdots\chi_{\mu_n}^{(n)}\chi_{\mu_1'}^{(1)}\chi_{\mu_2'}^{(2)}\cdots\chi_{\mu_n'}^{(n)}e^{-(1/2)\sum_{\mu\nu}(\chi_\mu^\nu)^2}. \quad (122)$$

This factorizes into a product of integrals for fixed ν,

$$J_F = \epsilon_{\mu_1\mu_2\cdots\mu_n}\epsilon_{\mu_1'\mu_2'\cdots\mu_n'}\int\prod_\mu\frac{d\chi_\mu^{(1)}}{\sqrt{2\pi}}\chi_{\mu_1}^{(1)}\chi_{\mu_1'}^{(1)}e^{-(1/2)\sum_\mu(\chi_\mu^{(1)})^2} \times \int\prod_\mu\frac{d\chi_\mu^{(2)}}{\sqrt{2\pi}}\chi_{\mu_1}^{(2)}\chi_{\mu_1'}^{(2)}e^{-(1/2)\sum_\mu(\chi_\mu^{(2)})^2}\cdots \times \int\prod_\mu\frac{d\chi_\mu^{(n)}}{\sqrt{2\pi}}\chi_{\mu_1}^{(n)}\chi_{\mu_1'}^{(n)}e^{-(1/2)\sum_\mu(\chi_\mu^{(n)})^2}.$$

Each integral in the product is equal to 1 except for those giving a δ function with unit coefficient:

$$J_F = \epsilon_{\mu_1\mu_2\cdots\mu_n}\epsilon_{\mu_1'\mu_2'\cdots\mu_n'}\delta_{\mu_1,\mu_1'}\delta_{\mu_2,\mu_2'}\cdots\delta_{\mu_n,\mu_n'} = \epsilon^2_{\mu_1\mu_2\cdots\mu_n} = n\,!\,. \quad (123)$$

We have then

$$n_0 P[\mathbf{V},t] = n\,!\prod_\mu\left(\frac{S_\mu^{(2)}}{2\pi\sqrt{S_0\bar{S}}}\right)\frac{1}{(1+\tilde{V}^2)^{(n+2)/2}}. \quad (124)$$

Since $P(\mathbf{V},t)$ is normalized to 1, we find on integration over $\mathbf{V}$ the result

$$n_0 = \frac{n!}{2^{n/2}\Gamma\left(\frac{n}{2}+1\right)}\prod_\mu\sqrt{\frac{S_\mu^{(2)}}{2\pi S_0}} \quad (125)$$

which agrees with previous results in the isotropic limit. Eliminating n_0 in Eq. (124) we obtain the final result for the VVPDF:

$$P[\mathbf{V},t] = \frac{\Gamma\left(\frac{n}{2}+1\right)}{\pi^{n/2}}\left(\prod_{\mu}\frac{1}{\overline{v_\mu}}\right)\frac{1}{(1+\tilde{V}^2)^{(n+2)/2}}, \quad (126)$$

where $\tilde{V}_\alpha = V_\alpha/\overline{v_\alpha}$,

$$\overline{v_\mu} = \sqrt{\frac{\overline{S}}{S_\mu^{(2)}}}, \quad (127)$$

using Eqs. (77), (78), and (81) we have

$$\overline{S} = S_0\frac{1}{4}\left[\frac{\ddot{I}}{I} - \left(\frac{\dot{I}}{I}\right)^2\right], \quad (128)$$

while Eq. (79) gives

$$S_\mu^{(2)} = S_0\frac{1}{I}\int\frac{d^dq}{(2\pi)^d}q_\mu^2 e^{-2O(q)(t-t_0)}f(\mathbf{q},t_0) \quad (129)$$

with I defined by Eq. (51):

$$I = \int\frac{d^dq}{(2\pi)^d}e^{-2O(q)(t-t_0)}f(\mathbf{q},t_0). \quad (130)$$

The needed input to determine the average vortex speed $\overline{v_\mu}$ is the function $O(\mathbf{q})$, the Fourier transform of the operator defined by Eq. (3), and the initial condition $f(\mathbf{q},t_0)$. This result for $P[\mathbf{V},t]$ is the anisotropic generalization of Eq. (1). For the set of models included here the velocity tail exponent is $(n+2)$ independent of direction.

VI. ANISOTROPIC CASE

As a particularly simple example, suppose that we have the choice in the governing Langevin equation

$$\hat{O}(1)\psi_\nu(1) = -\sum_\alpha c_\alpha\nabla_\alpha^2\psi_\nu(1), \quad (131)$$

or, in terms of Fourier transforms,

$$O(\mathbf{q}) = \sum_\alpha c_\alpha q_\alpha^2. \quad (132)$$

The associated vortex-velocity probability distribution is given by Eq. (126) and we need to work out the average vortex speed given by Eq. (127). Assuming the initial condition

$$f(q,0) = \left(\prod_\alpha(2\pi h_\alpha)^{1/2}\right)e^{-(1/2)h_\mu q_\mu^2}, \quad (133)$$

we have from Eq. (130)

$$I(t) = \int\frac{d^nq}{(2\pi)^n}\left(\prod_\alpha(2\pi h_\alpha)^{1/2}\right)e^{-(1/2)h_\mu(t)q_\mu^2} = \prod_\alpha\left(\frac{h_\alpha}{h_\alpha(t)}\right)^{1/2}, \quad (134)$$

where

$$h_\alpha(t) = h_\alpha + 4c_\alpha(t-t_0), \quad (135)$$

from Eq. (128)

$$\frac{\overline{S}}{S_0} = 2\sum_\mu\left(\frac{c_\mu}{h_\mu(t)}\right)^2, \quad (136)$$

and from Eq. (129)

$$\frac{S_\mu^{(2)}(1)}{S_0(1)} = \int\frac{d^nq}{(2\pi)^n}q_\mu^2 f(q,t_1) = \frac{1}{h_\mu(t)}. \quad (137)$$

Putting these results back into Eq. (127) we find the scaling velocity for a simple anisotropic system is given by

$$\overline{v_2^\mu} = 2h_\mu(t)\sum_\alpha\left(\frac{c_\alpha}{h_\alpha(t)}\right)^2 \quad (138)$$

$$= 2[h_\mu + 4c_\mu(t-t_0)]\sum_\alpha\left(\frac{c_\alpha}{h_\alpha+4c_\alpha(t-t_0)}\right)^2. \quad (139)$$

In the large time limit we have

$$\overline{v_\mu^2} = \frac{d}{2}\frac{c_\mu}{t} \quad (140)$$

and the final form is a simple generalization of the isotropic result.

VII. CONSERVED ORDER PARAMETER CASE

Let us look at the COP case where $O(q) = -q^2 + q^4$. We choose the rather general initial condition

$$f(q,0) = \left(\frac{h}{2\pi}\right)^{d/2}e^{-(h/2)q^2}, \quad (141)$$

which satisfies the normalization condition given by Eq. (58). We then need to evaluate the integral I and the numerator in Eq. (129):

$$J = \int\frac{d^dq}{(2\pi)^d}\frac{q^2}{d}e^{2t(q^2-q^4)}\left(\frac{h}{2\pi}\right)^{d/2}e^{-(h/2)q^2} = -\frac{2}{d}h^{d/2}\frac{\partial}{\partial h}(h^{-d/2}I). \quad (142)$$

We see that all of the ingredients contributing to $\overline{v^2}$ can be expressed in terms of I and its derivatives. We see that I can be written in the form

$$I = \tilde{I}_0 h^{d/2}\int_0^\infty dq\, q^{d-1}e^{2(q^2-q^4)t-(h/2)q^2}, \quad (143)$$

where $\tilde{I}_0$ is a constant which depends on d and cancels when we take ratios. Changing integration variables to $x = q^2$ we find

$$I = \tilde{I}_0' h^{d/2}\int_0^\infty dx\, x^{d/2-1}e^{2(x-x^2)t-(h/2)x}, \quad (144)$$

where $\tilde{I}_0' = \tilde{I}_0/2$. The leading large time dependence can be extracted from the integral by completing the square in the argument of the exponential or using the stationary phase method. We find, to leading order in large t

$$I=\tilde{I}_0\left(\frac{b}{2}\right)^{d/2-1}\sqrt{\frac{\pi}{2t}}e^{\phi^2}, \qquad (145)$$

where $b=1-h/4t$, $\phi^2=(t/2)b^2$. From this result for I we see that $\dot{I}=\omega I$ where

$$\omega=\left(\frac{d}{2}-1\right)\frac{\dot{b}}{b}+2\phi\dot{\phi}-\frac{1}{2t}=\frac{1}{2}-\frac{1}{2t}+\mathcal{O}(t^{-2}). \qquad (146)$$

Going further we have $\ddot{I}=\omega^2 I+\dot{\omega}I$ which leads easily to the useful result

$$\frac{\ddot{I}}{I}-\left(\frac{\dot{I}}{I}\right)^2=\dot{\omega}. \qquad (147)$$

Turning to the evaluation of J given by Eq. (142), using Eq. (145), we find

$$J=\frac{1}{2dt}\left[\frac{(d/2-1)}{b}+2\phi\frac{\partial\phi}{\partial b}\right]I. \qquad (148)$$

Working to leading order in time we find $J=I/2d$. Putting all of this together in Eq. (127),

$$\overline{v^2}=\frac{\dot{\omega}}{J/I}=\frac{d}{t^2} \qquad (149)$$

since, from Eq. (146), $\dot{\omega}=1/2t^2+\mathcal{O}(t^{-3})$. The final result for $\overline{v^2}$ is independent of the initial conditions. We see that the COP average vortex speed is qualitatively slower than the NCOP case:

$$\frac{\overline{v^2_{COP}}}{\overline{v^2_{NCOP}}}\approx\frac{1}{t}. \qquad (150)$$

The computation of $\overline{v_\mu}$ using Eq. (127) has been checked numerically in the simplest $n=d=2$ case where I can be evaluated explicitly in terms of an erfc function.

VIII. CONCLUSIONS

We have presented here the detailed calculation of the VVPDF including the time dependent vortex scaling velocity $\overline{v_\mu}$ for a class of models beyond the original nonconserved TDGL models. The class of models studied includes the conserved TDGL model and certain anisotropic models. In the conserved case it is found that the average vortex speed falls off as t^{-1} compared to the NCOP case where $\overline{v}\approx t^{-1/2}$. It is our intension to numerically test the predictions for the COP case.

We see that there is self-consistent confirmation that in dealing with vortex velocities one can organize things in terms of averages over an auxiliary Gaussian field. We require self-consistently that this field and the order parameter field share the same zeros. A similar development can be worked out for string defects [5,6].

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